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Monotone and Comonotone Polynomial Approximation Revisited

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Using a suitable Peetre functional that weighs differently the behaviour of the function in the middle of the interval and near the endpoints, we obtain estimates of the Jackson type on the rate of monotone polynomial approximation to a monotone continuous function. These estimates involve the second modulus of smoothness related to the Peetre functional. Then we apply these estimates to get estimates on the degree of comonotone polynomial approximation of a piecewise monotone function and on the degree of copositive polynomial approximation of a continuous function that changes sign finitely often in the interval.

1. INTRODUCTION AND MAIN RESULTS

Jackson type estimates for the approximation of montone functions $f \in C^k[-1, 1]$ by monotone polynomials have been known for more than a decade. These estimates are due to Lorentz and Zeller [10] for $k=0$, to Lorentz [9] for $k=1$, and to DeVore [2] for $k \geqslant 2$, and can be summarized as follows. For each $k \geqslant 0$ and every monotone nondecreasing $f \in C^k[-1, 1]$ there are nondecreasing polynomials p_n of degree not exceeding n such that

$$\| f - p_n \|_\infty \leqslant C n^{-k} \omega(f^{(k)}, 1/n), \tag{1}$$

where C is an absolute constant independent of f and n, and ω is the usual modulus of continuity of $f^{(k)}$.

If a function $f \in C[-1, 1]$ is piecewise monotone, i.e., changes monotonicity finitely many times in $[-1, 1]$, we say that a polynomial p_n is comonotone with f if p_n is piecewise monotone and changes its

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monotonicity exactly where f does. Estimates of the form (1) on comonotone approximation are known only for $k=0$ (Newman [11] and Iliev [7]) and for $k=1$ (Beatson and Leviatan [1]). It should be stressed that the constant C here depends on the number of changes of monotonicity of f but not on the location of these changes.

Recently DeVore and Yu [3] gave a constructive proof of some pointwise estimates of the Timan–Teljakovski type for monotone polynomial approximation. Specifically they proved that for a nondecreasing $f \in C[-1, 1]$ there are nondecreasing polynomials p_n for which

$$|f(x) - p_n(x)| \leqslant C\omega_2(f, \sqrt{1-x^2}/n), \qquad -1 \leqslant x \leqslant 1. \tag{2}$$

Here again C is an absolute constant independent of f and n.

In the sequel, unless specifically stated otherwise, C denotes an absolute constant not necessarily the same on each occurrence even if it appears more than once in the same equation or inequality.

While estimate (2) is the proper one for obtaining inverse theorems in polynomial approximation and indeed is applied in [3] to characterize nondecreasing functions in Lip* α, $0<\alpha<2$, by mean of the order of monotone polynomial approximation, there are increasing functions strictly in Lip α, $0<\alpha<1$, whose order of non constrained approximation is better than the one guaranteed by (1) or (2). For instance, if $0<\alpha<1$ and $f(x) = (1+x)^{\alpha/2} - 1$, $-1 \leqslant x < 0$, and $f(x) = 1-(1-x)^{\alpha/2}$, $0 \leqslant x \leqslant 1$, then f is approximable at the rate of $n^{-\alpha}$ while (1) and (2) (except near the endpoints) yield $n^{-\alpha/2}$. Our estimates below guarantee the rate $n^{-\alpha}$ in this case. Thus the rate of monotone approximation to the above $f(x)$ is $n^{-\alpha}$. We shall apply some of the ideas in [3] and obtain estimates involving a suitable Peetre functional, namely, for $f \in C[-1, 1]$ let

$$K_2(f, t) = \inf\{\|f-g\|_\infty + t^2 \|(1-x^2)\, g''(x)\|_\infty\}, \tag{3}$$

where the infimum is taken over all functions $g \in C^1[-1, 1]$ such that g' is locally absolutely continuous in $[-1, 1]$ and $(1-x^2)\, g''(x) \in L_\infty[-1, 1]$.

Our main result on monotone approximation is

THEOREM 1. *There exists an absolute constant C such that for every $f \in C[-1, 1]$ monotone nondecreasing and every $n \geqslant 1$ there is a nondecreasing polynomial of degree $\leqslant n$ such that*

$$\|f - p_n\| \leqslant CK_2(f, 1/n). \tag{4}$$

Evidently Theorem 1 yields global estimates on the degree of monotone approximation by polynomials, hence (4) is different from (2).

Nevertheless, the K-functional intrinsically involves $1-x^2$, thus taking into account the distance to the endpoints of the interval.

Indeed Ditzian [4, 5] shows that this K-functional is equivalent to a suitable modulus of smoothness, which again takes into account the distance to the endpoints of the interval. Namely, let $\varphi(x)=\sqrt{1-x^2}$ and define

$$\omega_2^\varphi(f, t)=\sup_{0\leqslant h\leqslant t}\|\Delta^2_{h\varphi(x)}f(x)\|_\infty,$$

where

$$\begin{aligned}\Delta^2_{h\varphi(x)}f(x)&=f(x-h\varphi(x))-2f(x)+f(x+h\varphi(x)), && x\pm h\varphi(x)\in[-1,1]\\ &=0, && \text{otherwise.}\end{aligned}$$

Then Ditzian ([4, Theorem 3.1] or [5, Theorem A]) proved that

$$C_1\omega_2^\varphi(f, t)\leqslant K_2(f, t)\leqslant C_2\omega_2^\varphi(f, t).$$

Therefore we can rewrite Theorem 1 in a somewhat more familiar form.

THEOREM 1′. *Let $f\in C[-1, 1]$ be nondecreasing in $[-1, 1]$. Then for each $n\geqslant 1$ there is a nondecreasing polynomial p_n such that*

$$\|f-p_n\|_\infty\leqslant C\omega_2^\varphi(f, 1/n). \tag{4′}$$

It should be noted that Ditzian [5, Theorem 3.1] proves a similar estimate for nonconstrained polynomial approximation. Furthermore, in a yet unpublished result Ditzian and Totik [6] obtain inverse theorems relating ω_2^φ and the rate of polynomial approximation, thus enabling one to retrieve information on $\omega_2^\varphi(f,\cdot)$ from the degree of polynomial approximation to f.

The author is indebted to Z. Ditzian for discussing with him those unpublished results.

Similar to the above one can define, for $f\in C[-1, 1]$,

$$K_1(f, t)=\inf\{\|f-g\|_\infty+t\,\|\sqrt{1-x^2}\,g'(x)\|_\infty\}, \tag{5}$$

where the infimum is taken over all $g\in C[-1, 1]$ which are locally absolutely continuous in $[-1, 1]$ and $\sqrt{1-x^2}\,g'(x)\in L_\infty[-1, 1]$. Equivalently, for $f\in C[-1, 1]$ let

$$\omega^\varphi(f, t)=\sup_{0\leqslant h\leqslant t}\|\Delta_{h\varphi(x)}f(x)\|_\infty$$

with $\varphi(x)=\sqrt{1-x^2}$, where

$$\Delta_{h\varphi(x)} f(x)=f\left(x-\frac{h}{2}\varphi(x)\right)-f\left(x+\frac{h}{2}\varphi(x)\right), \qquad x\pm\frac{h}{2}\varphi(x)\in[-1,1]$$
$$=0, \qquad \text{otherwise.}$$

It is not difficult to see that $\omega_2^\varphi(f,t)\leqslant C\omega^\varphi(f,t)$.

Again by [5, Theorem A] we have

$$C_1\omega^\alpha(f,t)\leqslant K_1(f,t)\leqslant C_2\omega^\varphi(f,t). \tag{6}$$

Now it is readily seen that for $f\in C^1[-1,1]$,

$$\Delta_{h\varphi}^2 f(x)=\varphi(x)\int_0^h \Delta_{2v\varphi(x)} f'(x)\,dv.$$

Hence $\omega_2^\varphi(f,t)\leqslant t\omega^\varphi(f',2t)\leqslant Ct\omega^\varphi(f',t)$.

Thus an immediate consequence of Theorem 1 is

COROLLARY 2. *Let $f\in C^1[-1,1]$ be nondecreasing in $[-1,1]$. Then for each $n\geqslant 1$ there is a nondecreasing polynomial p_n such that*

$$\|f-p_n\|\leqslant Cn^{-1}\omega^\varphi(f',n^{-1}). \tag{7}$$

In fact we will show a little more, namely,

THEOREM 3. *Let $f\in C^1[-1,1]$ be nondecreasing in $[-1,1]$. Then for each $n\geqslant 1$, there is a nondecreasing polynomial p_n such that (7) holds and*

$$\|f'-p_n'\|_\infty\leqslant C\omega^\varphi(f',n^{-1}). \tag{8}$$

This will enable us to extend the result to piecewise monotone functions. To this end we have

THEOREM 4. *Let $f\in C^j[-1,1]$, $j=0$ or 1, have $r\geqslant 1$ changes of monotonicity in $[-1,1]$ and let α be the point of change of monotonicity closest to the endpoints. Then for each $n\geqslant 1$ there is a polynomial p_n comonotone with f such that*

$$\|f-p_n\|\leqslant C_r(\alpha)\,n^{-j}\omega^\varphi(f^{(j)},n^{-1}) \tag{9}$$

and if $j=1$ also

$$\|f'-p_n'\|\leqslant C_r(\alpha)\,\omega^\varphi(f',n^{-1}). \tag{10}$$

Here $C_r(\alpha)$ is an absolute constant depending on r and α but is otherwise independent of f and of n. In fact $C_r(\alpha)\leqslant C_r/\varphi(\alpha)^r$.

Last, an immediate consequence of Theorem 4 is an estimate on the approximation of $f \in C[-1, 1]$ which changes its sign finitely many times in $[-1, 1]$ by means of polynomials p_n which are copositive with f on $[-1, 1]$, i.e., $f(x)\, p_n(x) \geqslant 0$, $-1 \leqslant x \leqslant 1$. (Compare with [8].)

COROLLARY 5. *Let $f \in C[-1, 1]$ have r changes in sign in $[-1, 1]$ and let α be the point of change of sign closest to the endpoints. Then for each $n \geqslant 1$, there is a polynomial copositive with f such that*

$$\| f - p_n \| \leqslant C_r(\alpha)\, \omega^{\varphi}(f, n^{-1}). \tag{11}$$

2. Monotone Polynomial Approximation

Following [3] we approximate f by a piecewise linear function S_n which interpolates f at certain points $-1 = \xi_{-n} < \xi_{-n+1} < \cdots < \xi_n = 1$, to be described later. Thus in $[\xi_j, \xi_{j+1}]$, S_n has the slope

$$s_j = \frac{f(\xi_{j+1}) - f(\xi_j)}{\xi_{j+1} - \xi_j} = f[\xi_j, \xi_{j+1}], \qquad j = n, ..., n-1,$$

and if $\varphi_j(x) = (x - \xi_j)_+$ we can write

$$S_n(x) = f(-1) + s_{-n}(1 + x) + \sum_{j=-n+1}^{n-1} (s_j - s_{j-1})\, \varphi_j(x).$$

Now it follows by Newton's formula that

$$f(x) - S_n(x) = f[\xi_j, x, \xi_{j+1}](x - \xi_j)(x - \xi_{j+1}), \quad \xi_j \leqslant x \leqslant \xi_{j+1}, \tag{12}$$

where the square brackes denote the divided difference of f at ξ_j, x, ξ_{j+1}.

The choice of the ξ_j's is made in the following way. Let $J_n(t)$ denote the Jackson kernel

$$J_n(t) = \lambda_n \left(\frac{\sin nt/2}{\sin t/2}\right)^8, \qquad \int_{-\pi}^{\pi} J_n(t)\, dt = 1,$$

and define

$$T_j(t) = \int_{t-t_j}^{t+t_j} J_n(u)\, du, \qquad j = 0, 1, ..., 2n,$$

where $t_j = j\pi/2n$, $j = 0, 1, ..., 2n$. Note that in particular $T \equiv 0$ and $T_{2n} \equiv 1$.

Now for $x = \cos t$ let $r_j(x) = T_{n-j}(t)$ and define

$$R_j(x) = \int_{-1}^{x} r_j(u)\, du, \qquad j = -n, ..., n.$$

Note again that in particular $R_{-n}(x) = 1 + x$ and $R_n(x) \equiv 0$. The points ξ_j are defined by the equations (see [3])

$$1 - \xi_j = R_j(1)$$

and since $R_j - R_{j+1}$ is nonnegative and increasing in $[-1, 1]$ we get $-1 = \xi_{-n} < \xi_{-n+1} < \cdots < \xi_n = 1$.

We shall see that $R_j(x)$ is a sufficiently good approximation to $\varphi_j(x) = (x - \xi_j)_+$ to guarantee estimates (4) and (8).

Define the operator L_n by

$$\begin{aligned} L_n(f) &= f(-1) + s_{-n} R_{-n} + \sum_{j=-n+1}^{n-1} (s_j - s_{j-1}) R_j \\ &= f(-1) + \sum_{j=-n}^{n-1} s_j (R_j - R_{j+1}) \end{aligned}$$

and it follows that $L_n(f)$ is nondecreasing in $[-1, 1]$ if f is. Also it was shown by DeVore and Yu [3] that $\|L_n\|$ $(n \geqslant 1)$ are uniformly bounded ($\|L_n\|$ is the operator norm of L_n: $C[-1, 1] \to [-1, 1]$).

In the course of our proof we need the following information on the location of the ξ_j's (see [3]).

LEMMA A. *Let $\delta_j = (\sin t_{n-j})/n + 1/n^2$, $j = -n, ..., n$. Then we have*

(i) $c_0 \delta_j \leqslant \xi_{j-1} - \xi_j \leqslant c_1 \delta_j$, $j = -n, ..., n-1$,

(ii) $c_0 \delta_j \leqslant \delta_{j+1} \leqslant c_1 \delta_j$, $j = -n, ..., n-1$.

(iii) *For any* $\xi_j \leqslant u \leqslant \xi_{j+1}$, $-n+1 \leqslant j \leqslant n-2$,

$$\xi_{j+1} - \xi_j \leqslant c \frac{\sqrt{1-u^2}}{n}.$$

Last, put

$$d_j(t) = \max\{1; n \operatorname{dist}(t, \{t_j, -t_j\})\},$$

then DeVore and Yu [3, (2.12)(ii)] showed that for $x = \cos t$, $0 \leqslant t \leqslant \pi$, we have

$$|\varphi_j'(x) - R_j'(x)| \leqslant C(d_{n-j}(t))^{-7}, \qquad x \neq \xi_j,\ j = -n, ..., n-1. \tag{13}$$

We prove the following lemma (compare with [3, (2.12)(i)]).

LEMMA 6. *For* $j=-n+1, ..., n-1$, $x=\cos t$, $0\leqslant t\leqslant \pi$, we have

$$|\varphi_j(x)-R_j(x)|\leqslant C\frac{\sin t_{n-j}}{n}(d_{n-j}(t))^{-5}. \tag{14}$$

Proof. Start with

$$\begin{aligned}\varphi_j(x)-R_j(x)&=\int_{-1}^{x}[\varphi_j'(y)-R_j'(y)]\,dy\\ &=\int_{x}^{1}[\varphi_j'(y)-R_j'(y)]\,dy\end{aligned} \tag{15}$$

since $\varphi_j(\pm 1)=R_j(\pm 1)$, $j=-n+1, ..., n-1$.

If $t\geqslant t_{n-j}$, then using the left equality in (15) and (13) we get

$$\begin{aligned}|\varphi_j(x)-R_j(x)|&\leqslant\int_{-1}^{x}|\varphi_j'(y)-R_j'(y)|\,dy\\ &\leqslant C\int_{t}^{\pi}\sin u(d_{n-j}(u))^{-7}\,du.\end{aligned}$$

Now $\sin u\leqslant \sin t_{n-j}+|u-t_{n-j}|$ and the proof follows by the inequality [3, (2.14)(ii)], for $k=0, 1$,

$$\int_{t}^{\pi}|u-t_j|^k\,(d_j(u))^{-7}\,du\leqslant Cn^{-k-1}(d_j(t))^{-5},\qquad t_j\leqslant t\leqslant\pi,$$

and the observation that $\sin t_{n-j}\geqslant \sin t_1\geqslant 1/n$. If $t<t_{n-j}$ we use the right equality in (15) and proceed in a similar way.

We are ready now to state and prove a special case of Theorem 1 where f is twice differentiable.

THEOREM 7. *Let* $f'\in C[-1,1]$ *be locally absolutely continuous and assume* $|(1-x^2)\,f''(x)|\leqslant M$ *a.e. in* $[-1,1]$. *Then for each* $n\geqslant 1$ *we have*

$$\|f-L_n(f)\|\leqslant CM/n^2. \tag{16}$$

Proof. We will estimate $\|f-S_n\|$ and $\|S_n-L_n(f)\|$. By (12), for $\xi_j\leqslant x\leqslant \xi_{j+1}$,

$$|f(x)-S_n(x)|\leqslant|f[\xi_j,x,\xi_{j+1}]|\,(x-\xi_j)(\xi_{j+1}-x).$$

Now

$$f(t)=f(\xi_j)+(t-\xi_j)\,f'(\xi_j)+\int_{\xi_j}^{1}(t-u)_+\,f''(u)\,du,$$

which implies for $\xi_j\leqslant x\leqslant\xi_{j+1}$ that

$$f[\xi_j,x,\xi_{j+1}]=\frac{1}{\xi_{j+1}-\xi_j}\left[\int_{\xi_j}^{x}\frac{u-\xi_j}{x-\xi_j}f''(u)\,du+\int_{x}^{\xi_{j+1}}\frac{\xi_{j+1}-u}{\xi_{j+1}-x}f''(u)\,du\right].$$

Hence for $\xi_j\leqslant x\leqslant\xi_{j+1}$

$$\begin{aligned}|f(x)-S_n(x)|&\leqslant\frac{1}{\xi_{j+1}-\xi_j}\left[\int_{\xi_j}^{x}(u-\xi_j)(\xi_{j+1}-x)\,|f''(u)|\,du\right.\\&\qquad\left.+\int_{x}^{\xi_{j+1}}(\xi_{j+1}-u)(x-\xi_j)\,|f''(u)|\,du\right]\\&\leqslant\frac{1}{\xi_{j+1}-\xi_j}\int_{\xi_j}^{\xi_{j+1}}(u-\xi_j)(\xi_{j+1}-u)\,|f''(u)|\,du.\end{aligned}$$

If $-n+1\leqslant j\leqslant n-2$ it follows by Lemma A(iii) that

$$\begin{aligned}|f(x)-S_n(x)|&\leqslant\frac{C}{\xi_{j+1}-\xi_j}\int_{\xi_j}^{\xi_{j+1}}\frac{1-u^2}{n^2}\,|f''(u)|\,du\\&\leqslant CM/n^2.\end{aligned}\tag{17}$$

For $j=-n$ or $j=n-1$ we have by virtue of Lemma A(iii) that

$$(u-\xi_{-n})(\xi_{-n+1}-u)\leqslant C(1+u)\,\delta_{-n}\leqslant C\frac{1+u}{n^2}\leqslant C\frac{1-u^2}{n^2},\quad -1\leqslant u\leqslant\xi_{-n+1}$$

$$(u-\xi_{n-1})(\xi_n-u)\leqslant C(1-u)\,\delta_n\leqslant C\frac{1-u}{n^2}\leqslant C\frac{1-u^2}{n^2},\qquad \xi_{n-1}\leqslant u\leqslant 1,$$

and again (17) holds. Thus

$$\|f-S_n\|\leqslant CM/n^2,\tag{18}$$

and we have to estimate

$$S_n(x)-L_n(f)(x)=\sum_{j=-n+1}^{n-1}(s_j-s_{j-1})(\varphi_j(x)-R_j(x)).$$

By Lemma 1 and the readily seen estimate

$$\sum_{j=-n+1}^{n-1} (d_{n-j}(t))^{-5} = O(1),$$

it suffices to show that

$$|s_j - s_{j-1}| \frac{\sin t_{n-j}}{n} \leqslant CM/n^2, \qquad -n+1 \leqslant j \leqslant n-1. \tag{19}$$

To this end note that, as above, we have

$$\begin{aligned} |s_j - s_{j-1}| &= |f[\xi_{j-1}, \xi_j, \xi_{j+1}]| \, (\xi_{j+1} - \xi_j) \\ &\leqslant \int_{\xi_{j-1}}^{\xi_j} \frac{u - \xi_{j-1}}{\xi_j - \xi_{j-1}} |f''(u)| \, du \\ &\quad + \int_{\xi_j}^{\xi_{j+1}} \frac{\xi_{j+1} - u}{\xi_{j+1} - \xi_j} |f''(u)| \, du. \end{aligned} \tag{20}$$

By virtue of Lemma A, if $j \leqslant n-2$ and $\xi_j \leqslant u \leqslant \xi_{j+1}$, then

$$\begin{aligned} &\frac{\sin t_{n-j}}{n} \leqslant \delta_j \leqslant C(\xi_{j+1} - \xi_j) \leqslant C \frac{\sqrt{1-u^2}}{n} \\ &\xi_{j+1} - u \leqslant C \frac{\sqrt{1-u^2}}{n}. \end{aligned} \tag{21}$$

If $j \geqslant -n+2$ and $\xi_{j-1} \leqslant u \leqslant \xi_j$, then

$$\begin{aligned} &\frac{\sin t_{n-j}}{n} \leqslant C\, \delta_{j-1} \leqslant C(\xi_j - \xi_{j-1}) \leqslant C \frac{\sqrt{1-u^2}}{n} \\ &u - \xi_{j-1} \leqslant C \frac{\sqrt{1-u^2}}{n}. \end{aligned} \tag{22}$$

If $j = n-1$ and $\xi_{n-1} \leqslant u \leqslant 1$, then

$$\begin{aligned} &\frac{\sin t_1}{n} = \frac{\sin(\pi/2n)}{n} \leqslant \frac{C}{n^2} \\ &\xi_n - u = 1 - u \leqslant 1 - u^2, \end{aligned} \tag{23}$$

and finally if $j = -n+1$, $-1 \leqslant u \leqslant \xi_{-n+1}$, then

$$\begin{aligned} &\frac{\sin t_{2n-1}}{n} = \frac{\sin(\pi/2n)}{n} \leqslant \frac{C}{n^2} \\ &u - \xi_{-n} = 1 + u \leqslant 1 - u^2. \end{aligned} \tag{24}$$

Plugging (21) through (24) in (20) we have

$$|s_j - s_{j-1}| \frac{\sin t_{n-j}}{n} \leqslant \frac{C}{\xi_j - \xi_{j-1}} \int_{\xi_{j-1}}^{\xi_j} \frac{1-u^2}{n^2} |f''(u)| \, du + \frac{C}{\xi_{j+1} - \xi_j} \int_{\xi_j}^{\xi_{j+1}} \frac{1-u^2}{n^2} |f''(u)| \, du \leqslant CM/n^2.$$

This proves (19) and implies

$$\|S_n - L_n(f)\| \leqslant CM/n^2,$$

which together with (18) proves (16) and concludes our proof.

We are ready to prove Theorem 1.

Proof of Theorem 1. By (3) there eixsts a function $g \in C^1[-1, 1]$ such that g' is locally absolutely continuous and $(1-x^2)\, g''(x) \in L_\infty[-1, 1]$, which satisfies

$$\|f - g\|_\infty \leqslant K_2(f, 1/n) \tag{25}$$

$$\|(1-x^2)\, g''(x)\|_\infty \leqslant n^2 K_2(f, 1/n). \tag{26}$$

Then by (16) and (26)

$$\begin{aligned} \|f - L_n(f)\|_\infty &\leqslant \|f - g\|_\infty + \|g - L_n(g)\|_\infty + \|L_n(f-g)\|_\infty \\ &\leqslant (1 + \|L_n\|)\, \|f - g\|_\infty + CK_2(f, 1/n) \end{aligned}$$

and since $\|L_n\|$ is uniformly bounded we get by (25) that

$$\|f - L_n(f)\|_\infty \leqslant CK_2(f, 1/n).$$

It was remarked already that if f is nondecreasing so is $L_n(f)$, which is a polynomial of degree not exceeding $4n$, so our proof is complete.

3. Comonotone Polynomial Approximation

We begin this section by proving Theorem 3, which provides simultaneously monotone approximation to f and approximation to f'. This control over the rate of approximation to f' will enable us to obtain the estimates on the comonotone approximation following the ideas of Beatson and the author [1].

Proof of Theorem 3. We only need to prove that for a nondecreasing $f \in C^1[-1, 1]$,

$$\| f' - L_n(f)' \|_\infty \leqslant C\omega^\varphi(f', 1/n).$$

To this end observe that we are done once we prove the following

PROPOSITION 8. *For* $f \in C[-1, 1]$, *let*

$$s_j = \frac{1}{\xi_{j+1} - \xi_j} \int_{\xi_j}^{\xi_{j+1}} f(u)\, du, \qquad -n \leqslant j \leqslant n-1$$

and define

$$L_n'(f)(x) = s_{-n} + \sum_{j=-n+1}^{n-1} (s_j - s_{j-1})\, R_j'(x).$$

Then

$$\| f - L_n'(f) \|_\infty \leqslant CK_1(f, 1/n). \tag{27}$$

Proof. Put $S_n'(x) = s_j$ for $\xi_j < x < \xi_{j+1}$, $j = -n, ..., n-1$. Then

$$S_n'(x) = s_{-n} + \sum_{j=-n+1}^{n-1} (s_j - s_{j-1})\, \varphi_j'(x), \quad x \neq \xi_j,\ j = -n, ..., n-1.$$

Assume first that f is locally absolutely continuous and that $|\sqrt{1-x^2}\, f'(x)| \leqslant M$ a.e. in $[-1, 1]$. We will show that

$$\| f - S_n' \|_\infty \leqslant CM/n \tag{28}$$

and

$$\| S_n' - L_n'(f) \|_\infty \leqslant CM/n. \tag{29}$$

Since this is done in much the same way as in proving (16) we will omit the details. Just observe that for $\xi_j < x < \xi_{j+1}$

$$\begin{aligned} f(x) - S_n'(x) &= f(x) - f(\xi_j) - \frac{1}{\xi_{j+1} - \xi_j} \int_{\xi_j}^{\xi_{j+1}} [f(u) - f(\xi_j)]\, du \\ &= \int_{\xi_j}^{x} f'(u)\, du - \frac{1}{\xi_{j+1} - \xi_j} \int_{\xi_j}^{\xi_{j+1}} (\xi_{j+1} - u)\, f'(u)\, du \\ &= \frac{1}{\xi_{j+1} - \xi_j} \left[\int_{\xi_j}^{x} (u - \xi_j)\, f'(u)\, du \right. \\ &\qquad \left. - \int_{x}^{\xi_{j+1}} (\xi_{j+1} - u)\, f'(u)\, du \right], \end{aligned}$$

so that

$$|f(x)-S_n'(x)| \leqslant \frac{C}{\xi_{j+1}-\xi_j} \int_{\xi_j}^{\xi_{j+1}} \frac{\sqrt{1-u^2}}{n} \times |f'(u)|\,du \leqslant CM/n$$

and (28) follows. And for $x \neq \xi_j$, $j=-n, ..., n-1$,

$$|S_n'(x)-L_n'(f)(x)| \leqslant \sum_{j=-n+1}^{n-1} |s_j - s_{j-1}|\,|\varphi_j'(x)-R_j'(x)|.$$

Now by virtue of (13) and the easy estimate

$$\sum_{j=-n+1}^{n-1} (d_{n-j}(t))^{-7} = O(1), \tag{30}$$

it suffices to show that

$$|s_j - s_{j-1}| \leqslant CM/n. \tag{31}$$

But

$$\begin{aligned} s_j - s_{j-1} &= \frac{1}{\xi_{j+1}-\xi_j} \int_{\xi_j}^{\xi_{j+1}} f(u)\,du \\ &\quad - \frac{1}{\xi_j-\xi_{j-1}} \int_{\xi_{j-1}}^{\xi_j} f(u)\,du \\ &= \frac{1}{\xi_{j+1}-\xi_j} \int_{\xi_j}^{\xi_{j+1}} [f(u)-f(\xi_j)]\,du \\ &\quad + \frac{1}{\xi_j-\xi_{j-1}} \int_{\xi_{j-1}}^{\xi_j} [f(\xi_j)-f(u)]\,du \\ &= \frac{1}{\xi_{j+1}-\xi_j} \int_{\xi_j}^{\xi_{j+1}} (\xi_{j+1}-u)\,f'(u)\,du \\ &\quad + \frac{1}{\xi_j-\xi_{j-1}} \int_{\xi_{j-1}}^{\xi_j} (u-\xi_{j-1})\,f'(u)\,du. \end{aligned}$$

Hence

$$\begin{aligned} |s_j - s_{j-1}| &\leqslant \frac{C}{\xi_{j+1}-\xi_j} \int_{\xi_j}^{\xi_{j+1}} \frac{\sqrt{1-u^2}}{n}\,|f'(u)|\,du \\ &\quad + \frac{C}{\xi_j-\xi_{j-1}} \int_{\xi_{j-1}}^{\xi_j} \frac{\sqrt{1-u^2}}{n}\,|f'(u)|\,du \leqslant CM/n. \end{aligned}$$

This proves (31) and completes the proof of (29). Thus

$$\| f - L_n'(f)\|_\infty \leqslant CM/n. \tag{32}$$

We will prove now that $\|L_n'\|$ is uniformly bounded. Let $f \in C[-1, 1]$. Then

$$|L_n'(f)(x)| \leqslant |S_n'(x)| + |L_n' f(x) - S_n'(x)|, \quad x \neq \xi_j \; j = -n, ..., n-1.$$

Now $|s_j| \leqslant \| f \|_\infty$, so it follows by (13) and (30) that

$$\begin{aligned} |L_n'(f)(x)| &\leqslant \| f \|_\infty + 2 \| f \|_\infty \sum_{j=-n+1}^{n-1} |\varphi'(x) - R_j'(x)| \\ &\leqslant \| f \|_\infty + C \| f \|_\infty \sum_{j=-n+1}^{n-1} (d_{n-j}(t))^{-7} \\ &\leqslant C \| f \|_\infty. \end{aligned}$$

Continuity of $L_n'(f)$ now assures

$$\|L_n'(f)\|_\infty \leqslant C \| f \|_\infty.$$

To complete the proof let $g \in C[-1, 1]$ be such that $\sqrt{1-x^2}\, g'(x) \in L_\infty[-1, 1]$ and

$$\| f - g\|_\infty \leqslant K_1(f, 1/n) \tag{33}$$

$$\|\sqrt{1-x^2}\, g'(x)\| \leqslant n K_1(f, 1/n). \tag{34}$$

Then

$$\begin{aligned} \| f - L_n'(f)\|_\infty &\leqslant \| f - g\|_\infty + \|g - L_n'(g)\|_\infty + \|L_n'(f-g)\|_\infty \\ &\leqslant (1 + \|L_n'\|)\, \| f - g\|_\infty + \|g - L_n(g)\|_\infty \\ &\leqslant C K_1(f, 1/n) \end{aligned}$$

by (32) through (34) and the uniform boundedness of $\|L_n'\|$.

Proof of Theorem 4. First observe that when estimating $f(x) - f(y)$ by means of $\omega^\varphi(f, |t|)$ we are looking for t so that $u - (t/2)\,\varphi(u) = x$ and $u + (t/2)\,\varphi(u) = y$. Thus $u = (x+y)/2$ and

$$t\varphi(u) = y - x$$

or

$$|t| = \frac{|y-x|}{\varphi(u)}.$$

Now

$$\begin{aligned}\varphi(u)=\sqrt{1-u^2}&=\sqrt{1-\left(\frac{y+x}{2}\right)^2}\\&=\frac{\sqrt{2-(y+x)}\sqrt{2+y+x}}{2}\\&=\tfrac{1}{2}\sqrt{(1-x)+(1-y)}\sqrt{(1+x)+(1+y)}\\&\geqslant\tfrac{1}{2}\max\{\varphi(x),\varphi(y)\}.\end{aligned}$$

We will prove the case $j=1$, the case $j=0$ being similar. For small n, say $n \leqslant N(r)$ (r the number of changes of monotonicity), the estimate is trivial. For let α be the point of change closest to the endpoints of the interval, then $f'(\alpha)=0$ and so the constant polynomial $p_n \equiv f(\alpha)$ approximates f as required because

$$\begin{aligned}|f(x)-f(\alpha)|&=|x-\alpha|\ |f'(\xi)|\\&=|x-\alpha|\ |f'(\xi)-f'(\alpha)|\\&\leqslant 2\omega^{\varphi}\left(f',\frac{2\ |\xi-\alpha|}{\varphi(\alpha)}\right)\\&\leqslant 2\omega^{\varphi}\left(f',\frac{4}{\varphi(\alpha)}\right)\end{aligned}$$

(by virtue of (6))

$$\begin{aligned}&\leqslant C\frac{n}{\varphi(\alpha)}\,\omega^{\varphi}\left(f',\frac{1}{n}\right)\\&\leqslant C\frac{N(r)^2}{\varphi(\alpha)}\,n^{-1}\omega^{\varphi}\left(f',\frac{1}{n}\right)\\&\leqslant C_r(\alpha)\,n^{-1}\omega^{\varphi}\left(f',\frac{1}{n}\right).\end{aligned}$$

Also

$$\begin{aligned}|f'(x)-0|&=|f'(x)-f'(\alpha)|\\&\leqslant C_r(\alpha)\,\omega^{\varphi}(f',1/n).\end{aligned}$$

So we have to prove the theorem for $n > N(r)$ and we do it by induction on r (the number of monotonicity changes). For $r=0$ this is Theorem 3, so let us assume (9) and (10) for functions with $r-1 \geqslant 0$ changes of monotonicity and $n \geqslant N(r-1)$ and prove we have them for f having r

changes of monotonicity. Given such an f with α the point of change closest to the endpoints, we may assume $f(\alpha)=0$ (otherwise subtract a constant). Define the "flipped" function

$$\begin{aligned} \hat{f}(x) &= f(x), \qquad x \geqslant \alpha \\ &= -f(x), \qquad x < \alpha. \end{aligned} \tag{35}$$

Then $\hat{f} \in C^1[-1, 1]$ and has only $r-1$ changes of monotonicity. We will show that

$$\omega^{\varphi}(\hat{f}', t) \leqslant C\omega^{\varphi}(f', t) \tag{36}$$

and by the induction hypothesis there is a polynomial p_n comonotone with $\hat{f}$ and satisfying (9) and (10). We therefore proceed as in the paper by Beatson and the author [1, proof of the lemma], where the only difference is that α plays the role of zero there and for $|x-\alpha| < k/n$ we have the inequalities

$$\begin{aligned} |\hat{f}'(x)| &= |\hat{f}'(x) - \hat{f}'(\alpha)| \\ &\leqslant \omega^{\varphi}\left(\hat{f}', \frac{2\,|x-\alpha|}{\varphi(\alpha)}\right) \\ &\leqslant C\frac{k}{\varphi(\alpha)}\,\omega^{\varphi}\left(\hat{f}', \frac{1}{n}\right) \end{aligned}$$

and

$$\begin{aligned} |\hat{f}(x)| &= |\hat{f}(x) - \hat{f}(\alpha)| \\ &= |x-\alpha|\;|\hat{f}'(\xi)| \\ &\leqslant C\frac{k^2}{n}\frac{1}{\varphi(\alpha)}\,\omega^{\varphi}\left(\hat{f}', \frac{1}{n}\right). \end{aligned}$$

Thus as in [1] we construct a polynomial P_{2n} which is comonotone with f and satisfies (9) and (10). We complete the proof by showing the validity of (36). In fact, by virtue of (6) it suffices to prove that for $f \in C[-1, 1]$ with $f(\alpha)=0$ and $\hat{f}$ given by (35) we have

$$K_1(\hat{f}, t) \leqslant CK_1(f, t).$$

To this end let $g \in C[-1, 1]$ be locally absolutely continuous and such that

$$\|f-g\|_{\infty} + t\,\|\sqrt{1-x^2}\,g'(x)\|_{\infty} \leqslant 2K_1(f, t).$$

Then we may assume without loss of generality that $g(\alpha)=0$. (For $|g(\alpha)| \leqslant 2K_1(f, t)$, thus taking $g_1 = g - g(\alpha)$ will do with the right-hand side being $CK_1(f, t)$.)

Now $\hat{g} \in C[-1, 1]$ and is locally absolutely continuous. Also

$$\|\hat{f} - \hat{g}\|_\infty = \|f - g\|_\infty$$

and

$$\|\sqrt{1-x^2}\, \hat{g}'(x)\|_\infty = \|\sqrt{1-x^2}\, g'(x)\|_\infty.$$

Hence

$$K_1(\hat{f}, t) \leqslant \|\hat{f} - \hat{g}\|_\infty + t\, \|\sqrt{1-x^2}\, \hat{g}'(x)\|_\infty \leqslant CK_1(f, t).$$

This concludes our proof.

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